# From Ancient Mathematics to Modern Technology

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#### Abstract

We start with the *arithmetic mean formula* and observe what a deep and profoundly useful meaning such a simple formula can have in car technology. Modern vehicles have their engine's power distributed to its right and left powering wheels by means of this simple formula. The mechanical device which does the trick is called a *differential* or *differential gear*. It allows the distribution of power to the right and left wheels so that the vehicle can drive straight or turn to either side. Elementary mathematical notions which we encounter while observing such a simple formula, have concrete and intuitive meanings in vehicle's motions. Comprehension of the meaning which such a simple mathematical formula provides, can also give an intuitive insight and motivation for the sophisticated concept of geodesics. We also present and discuss various other elementary and ancient mathematical ideas which have educationally wonderful and technologically useful applicability.

### 1 The arithmetic mean formula and car differential

We all know that cars are powered by motors. But how? How is the power (the rotation) of the motor transferred to the powering wheels which move the car? On a bicycle, we use a chain that transfers the rotation of the pedal to the back wheel. Is it not done very similarly in a car, just that the source of power is a

motor and not cyclist's muscles? Yes, but how. If the power of car's engine were transferred to the wheels just by simple rotation of the (say back) axis with left and right wheels attached to it, then both wheels would obviously rotate with the same rate (Figure 1). Being of the same size, both wheels would travel with the same speed.



Figure 1: Right and left wheels attached to the same axis of rotation

Describing the dependency of the three variables P, R and L by very intuitive means (for example, counting the rotations), we would have the trivial equalities

$$P = R = L.$$

But such a car could not turn, as it is obvious, that while turning, a car's left and right wheels move (rotate) with different speeds.

So how could the rotation be distributed to the right and to the left wheels so that the car could turn left or right? A sensible and real solution to the problem, which provides exactly the required "different distribution of power to the right and left wheels", comes in the form of the most simple of mathematical formulas, that is, the form of the *arithmetic mean formula*:

$$P = \frac{R+L}{2}.$$

The formula, which we do not know if and how it could be mechanically realized to power a car, simply says that "the right and left wheels should rotate in such a way, that its average remains equal to the power transmitted from the engine". This seems quite mysterious and surely gives a whole new perspective on the meaning, that such a simple formula can have. Namely, every modern car is powered via a so called *differential* or *differential gear*, which is nothing but a mechanical realization of the arithmetic mean formula.

Even though a car differential (differential gear) seems a rather sophisticated technical device, it can easily be explained with very elementary mathematical means. Imagine first that powering of the wheels is achieved by a "rotating handle on a disc", which is attached to the right and left discs, that are welded at the end of the right and left wheel axes, as shown in Figure 2.



Figure 2: Right, left and the powering discs

Instead of discs, we can imagine cogs. It is obvious, that with a help of such a mechanism, a rotation of "the handle P" would imply a "balanced rotation" of the left and right wheel, thus P = R = L. This seems far from the desired equation

$$P = \frac{R+L}{2},$$

but it is not. From the arithmetic mean formula we calculate

$$P-R=L-P=\frac{L-R}{2}.$$

Letting

$$X = \frac{L - R}{2},$$

we have

$$R = P - X$$
 and  $L = P + X$ 

Obviously, mathematically speaking, with the arithmetic mean formula, we have three variables and only one equation. Assuming (power) P is given, X can be seen as a free parameter, or within our "car engineering problem", the variable X tells us "how non straight" our driving is. If driving straight, we have X = 0. The sharper our right turn is, the bigger X is, and the right wheel rotation resistance increases while the left wheel rotation resistance decreases. The variable X describes "how much rotation" is transferred from the right wheel to the left. And the opposite occurs, if we turn left. The variable X, with this meaning, can be easily added to the sketch of the mechanism in Figure 2. Namely, if we allow that our "power disc" in Figure 2 freely rotates (free variable X) around the "handle", as indicated in Figure 3, we already have a model of a differential gear.



Figure 3: Right, left and freely revolvable powering discs

Once we understand the above explained idea, it is also easy to comprehend the functioning of a real car differential gear, which is illustrated in Figure 4. The power of engine is transformed through the rotation of "Cardan<sup>1</sup> driveshaft" and via a cog to the "revolving box" (see Figure 4), which literally realizes "our handle on a disc/cog". Note, that for the purpose of robustness of the mechanism, there are "two handles and two cogs" positioned on the opposite side (illustrated by dark gray in Figure 4).



Figure 4: Differential gear profile

The idea of a differential gear can be a useful didactical and motivational tool, especially as Lego models (see Figure 5) are easily accessible. By simple questions related to real life, we can give deeper understanding even to one of the simplest mathematical formulas. Assume we have a car with the engine turned off, no hand brakes applied, and in forward gear position. Knowing that

 $<sup>^1 {\</sup>rm named}$  after Italian mathematician Girolamo Cardano (1501 -1576)

"power/rotation of car's engine" (P) is related to the rotation of the left (L) and right (R) wheel by the formula  $P = \frac{R+L}{2}$ , we can ask two practical questions:

- a) Can the car be pushed forward (or backward)?
- b) Assume we have the car in the same position, but lifted up (as in a garage) with its wheels in the air. Can one of the powering wheels be moved/rotated by hand?

The two correct, and as experience shows for many, unexpected answers:

- a) No, the car can not be pushed forward (or backward) and
- b) yes, one powering wheel can easily be rotated (while the other rotates to the opposite direction)

are (as P = 0) elegantly conveyed by the formula R + L = 0.



Figure 5: Lego model of a car differential

## 2 Parallel paths, "straightness" and advanced mathematical concepts

After understanding, how cars are powered, it is intuitively quite obvious that vehicle's right and left wheels travel parallel paths: the axis (right and left half shafts) being perpendicular to the direction of the movement at each moment, while the wheels obviously remain equidistant throughout the drive. There are many interesting issues we can try to understand while observing parallel paths. One of the most interesting related problems and one with an incredibly simple and possibly unexpected answer is the problem of the comparison of the lengths of parallel paths ([7]).

The idea of comparing the lengths of parallel paths might be encountered also within a very different and simple (elementary geometry) question:

Assume the planet Earth is a perfect sphere and we put a ring around the equator which is 100 m longer than the equator. The ring is positioned equidistantly all around the equator. Is there enough space for a cat to slip through (under the ring)? What about for a mouse?

Contrary to intuitive expectations, the answer is affirmative: Evan a giraffe could easily walk under the ring. The answer is easily obtained via an equation from elementary geometry

$$2\pi(R+d) = 2\pi R + 100,$$

where R is the radius of the sphere (Earth) and d is the "distance" between the equator and the ring. Therefore

$$d = \frac{100 \text{ m}}{2\pi} \approx 16 \text{ m}$$

which is mathematically no surprise as the "radius" (R) and the "circumference" (c) of a circle are in linear correlation  $c = 2\pi R$ . We can see our equator and the ring as two parallel paths, which would be traveled by two wheels moving on an imaginary plane, the inside wheel along a circle (of equator size) and the outer at a distance d. The outer wheel would travel only for  $2\pi d$  longer distance than the inner wheel. And this  $2\pi d$  difference in the traveled path between the two wheels, is obviously independent of the size of the circle along which the inner wheel travels. That seems obvious for the two wheels traveling along any circle. What about if the parallel paths were more complicated?

It can be proved by the use of some (basic) differential geometry that the same result holds true for any (simple) closed parallel plane path (see Figure 6). Namely, for such paths the length of the outer path is exactly for  $2\pi d$  longer than the length of the inner path, regardless of their absolute length. And the result can be generalized even further. By sensible interpretations we can talk about the same result even for sectionally smooth (not closed) curves.



Figure 6: Random (simple) closed parallel path

Another interesting issue, which seems "straightforward", intuitively simple, but which after a thorough consideration opens up more questions than it offers answers, is the concept of "straightness".

We describe a very intuitive idea of *geodesics*, which might motivate its abstract mathematical definition much before we are able to comprehend its rigorous notion.

We have seen that vehicles have their engine's power (P) distributed to its right (R) and left (L) powering wheel by arithmetic mean formula  $P = \frac{R+L}{2}$ . The mechanical device which does the trick is called a *differential* or *differential qear.* It allows the distribution of power to the right and left wheel so that the vehicle can drive straight or turn to either side. By such a drive, a vehicle's right and left wheels might make sophisticated but parallel paths (on a plane). Differential gear also has a negative side, for example, if one is driving in snow or if a vehicle, say a tractor, is to pull a heavy load on an uneven terrain. Then a tractor might lean to one side, putting most of its weight on one wheel while the other one might slip and rotate freely without moving the vehicle. In such a case all the power would be transferred (for example) to the right wheel, while the left wheel would get none. In the language of our formula:  $P = \frac{R}{2}$  and L = 0. That is the reason why really powerful jeeps and tractors have an option to 'block' the differential gear, which, in the language of our formula, means P = R = L. But as said in the beginning, such a vehicle moves only on a straight path, as left and right wheels travel the same distance. OK, being on a plane, it is obvious that such a tractor moves straight, with its two wheels traveling along parallel (straight) lines. What about if such a tractor moves on a (smooth) wavy terrain? Then it would move along (an intuitive) geodesic, which is a generalization of the notion of a straight line. That is intuitively quite obvious: moving straight or by the shortest path to a desired destination means no "turns" are allowed. One can easily experiment with the idea by taking a simple toy of two wheels fixed on the same axes (for example, Lego) and rolling it on smooth surface (could be a sheet of hard paper rolled into a gutter) and trying to find 'shortest paths' between different points. Rolling such a toy, or driving a tractor with a blocked differential gear on a sphere surface, yields a movement along one of the great circles, which are known to be the "shortest paths" (geodesics) on a sphere.

#### 3 Ancient concept of a parabola and car lights

Simple geometric properties of a parabola, which were known already to the ancient Greeks explain the use of parabola in car lights and satellite dishes.

It is a nice intuitive approach in teaching the geometric concept of a parabola if we start with a computer simulation, where students can interactively 'play billiard' by shooting at a 'parabola shaped table' and trying to hit the (focus) point (see Figure 7).



Figure 7: Billiard on a parabola shaped table

Sooner rather than later students realize that horizontal launch results in the desired hit (interactive computer simulation can be reached at [3]). It is well known that such a statement can be easily proved by elementary geometry. Turning the direction of 'ball travel' and exchanging it with a beam of light, we can explore and easily explain car lights. It is interesting to first explore how a single beam of light reflects from a parabola (see Figure 8). Again, a computer simulation [4] is perfect to experiment.



Figure 8: Reflection of a beam on a parabola

Putting the source of light in the focus of a parabola, the beams reflected on a parabola travel a straight parallel path (see Figure 9). This explains car's long (head) lights.



Figure 9: Long car head lights – the source of light is in the focus

What about the short car head lights? Exploring the interactive computer simulation [4], we see that putting the source of light 'to the right' of the focus of a parabola gives a 'special reflection' which explains the short lights (see Figure 10).



Figure 10: The source of light is 'to the right' of the focus

Shading of the bottom part of beams creates the functionality of short car head lights (see Figure 11).



Figure 11: The source of light is 'to the right' of the focus

And that is exactly how the car lights function. Just a millimetre difference and a 'below-cap' under the 'short light wire' in a car light bulb makes all the difference between the long and short car lights (see Figure 12).



Figure 12: Long and short light wires in a car light bulb

It is also easy to explain, why the car light bulb has 'front metal cap'. The very similar ideas can be used to explain the functioning of a (parabola) satellite dish.

# 4 The concept of a discrete function and communication technology

It is intuitive that any sound, conversation, recording, ... can be presented as a function. We will not get into details of how a common sound is presented as a function. For example musicians know, that a perfectly sounding A - tone can be described by a function  $\sin(\pi 440t)$ , or one octave higher tone is presented by a function  $\sin(2\pi 440t)$ . As further examples one can present sounds expressed by functions like for example the following:

 $\sin(2\pi 440t)$   $\sin(34 + \sqrt{2}\sin(950t))$   $\sin(700t + 35t\sin(123t))$  $\sin(700t + \cos(150t) + 45t\sin(350t))$ 

All of the above sounds can be played for example by program Mathematica, where one has a command Play, which is very similar to the command Draw. Functions that describe 'artificial sounds' or pure tones (of a single frequency) are of 'orderly shapes' like sin function (see Figure 13).



Figure 13: Pure tone presented by a sin function

A simple human voice 'hello' is a much more complicated function (see Figure 14).



Figure 14: Recording of a human voice "HELLO" reveals much more complicated function

But whatever a sound, imagine now, that it is presented by a function. A true 'shape of a sound function' is not essential for our ideas. Thus, let us say that we could basically take any function to present a sound. Different functions would present different sounds. Let us start with a simple sin function.



Figure 15: Two sound presenting sin functions look very much the same

We draw two functions, which look very much the same (see Figure 15). Imagine that the 'top' function is a sound that is recorded on one side of a phone line and the 'bottom' function is a reproduced sound on the other side of the same phone line. Functions look exactly the same and it seems just to say, that phone line service provider is doing a good job, transmitting a perfect copy of the sound from one side to the other. But let us take literally a closer look and let us focus on both graphs at the point indicated by the arrow (see Figure 16).



Figure 16: A closer look at the two functions

Phone provider can 'cheat' and only transmit a discrete function, which consists of points at a certain distance. Of course, points have to be dense enough for customers, talking on the phone, not to notice any 'empty spaces'. Certainly, if the provider would only 'transmit' a point every five minutes, we would hear nothing. But if one imagines a point every millionth of a second, what we get is 'very smooth' looking function. Our above graphs (Figure 15) present the same sin function. The 'top' is an 'analogue' continuous function, while the 'bottom' one is a 'union of points'. What would the phone provider gain with such a cheating? Well, whenever we draw a point, we draw it of a certain thickness, but point's true thickness is 0. Imagine, we take further focus on the above point of the graph (see Figure 17).



Figure 17: An even closer look at our 'transmitted' function

It is now pretty obvious, why the phone line provider would 'cheat'. As the technology has long time ago won the race with human sensitivity and it can 'split time' to far tinier bits than a human ear could notice, we see, that technology offers a provider 'lots of free time'. The necessary density of points is determined by human ear sensitivity, and if the technology offers the 'split of time' to tenth of interval that human ear can notice, the machine can be programmed to 'listen' only one tenth of a time and is free nine tenths of a time. Basically, we see, that if we imagine the above 'dots' as discrete values of a function, we can squeeze ten other points in-between.

Let us look at the graph of four different functions (see Figure 18). We made the functions to intersect (at a point indicated by arrow) just to make it easier to explain our idea. It seems we have four different and 'precisely described functions'. It is clear that if we think of functions as sounds, this picture could easily be transformed (imagine color filter) into four different (clearly heard) sounds. And this is basically the trick of digital technology. People talking on a phone and real life users and customers of audio technology have very limited ear sensitivity and can be fooled to truly hear four different sounds from the signal visualized on the below picture. In fact we can say, that our eyes were fooled to see four different functions, while mathematically (that is precisely) speaking, we even do not have one function defined at the whole observed interval.



Figure 18: Visualization of the four different functions

To see and comprehend what we are talking about, let us again focus to the intersection point indicated by the arrow (see Figure 19).



Figure 19: A closer look ...

And an even closer focus reveals a profoundly different picture (see Figure 20).



Figure 20: A close-up view ...

We see, that 'much less than one digital' function can be made to carry enough information to reproduce 'four different functions', that are still precise enough to carry enough of the information necessary for human ear to 'hear a good sound'. Understanding the essence of functions and discretely defined functions, it is pretty obvious that this process looks unlimited. How many functions like that can be 'squeezed' within one discrete function? How many phone conversation can be squeezed into a single phone line? Of course it depends on the quality of the sound required (density of discrete points, which represent particular functions) and on the ability of technology to 'listen' and record' ever shorter bits of time. In reality, the machine would not only record bits of conversation on evenly spread out intervals, namely, that could result in recording noise (sound pollution), but the machine would record 'all the conversation' and transmit only (noise filtered) average in those tiny bits of time. With modern computer technology this wonderful and simple idea can easily be simulated by dynamic presentation of functions, when 'zoom in' and 'zoom out' can nicely and intuitively visualize how relative to human eve and ear a discrete or continuous looking functions can be. A nice interactive computer simulation of 'squeezing several functions within one' can be found at [5].

Finally, not as a complete joke, the idea can be given a funny but meaningful parallel. Imagine a class of students taking a test and a teacher attending the students (and taking care that students would not cheat). If a teacher leaves the classroom unattended, students might be tempted to start communicating and cheating. So it is hard to imagine, how the same teacher could take care of two different classes of students in two different classrooms at the same time. But that is because a teacher would be forced to leave at least one class unattended. But for how long? Imagine the teacher's strict eye is searching around the classroom every second... Theoretically, imagine the teacher who could shift its full presence and attention from one class to the other in tenth of a second. Is it not obvious that in such circumstances one such a speedy teacher could attend not only two but ten classes simultaneously?

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